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On the Plurality of Styles: *Spherics*, The Isoperimetric Problem

Abstract

This paper studies the question of plurality of mathematical styles, i.e., whether a fundamental mathematical work is characterised by a single style or by a multitude of styles; and, whether the unity of a subject in mathematics in its development is the outcome of a single style or several styles. The question is studied (a) within a single mathematical work and (b) through the study of the same problem over time and illustrated on Menelaus's *Sphaerica* and the isoperimetric problem.

Keywords

(Plurality of) mathematical styles, Gilles-Gaston Granger, the isoperimetric problem, the style of Menelaus's *Sphaerica*, the cosmological style, Al-Khāzin's geometric style, Ibn al-Haytham's infinitesimalistic style, the style of the calculus of variations, the style of synthetic geometry

Introduction

Historians of science agree that one of their main tasks is the reconstruction of scientific traditions. The task may seem easy because most traditions are represented by prominent names and distinctive features that make them recognisable. However, as soon as they are engaged in this task, they discover that it is a deceptive appearance that dissipates. Isn't it a characteristic

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of a scientific tradition to diversify and recreate itself according to the succession of the various authors and the rise of novel questions, thereby thwarting any reconstruction attempts?

During the last century, in their attempt to describe and analyse these facts, some philosophers of science have forged certain concepts, such as the *Denkform* by Ernst Cassirer (1874–1945), the "normal science" by Thomas Kuhn (1922–1996), the *épistémè* by Michel Foucault (1926–1984), and others.

Gilles-Gaston Granger (1920–2016), who had vast experience in the history of economics and the variety of schools that exercise it, deep knowledge of social mathematics going back to Marquis de Condorcet (1743–1794) and significant contribution to linguistics, has found in the concept of *style* a heuristic means to delineate traditions and carve out styles within a single tradition. This enables us to grasp the type of rationality that characterises each style. Indeed, it was due to the concept of style, which has proven valuable in literature and art history. Behind the variety of forms and mutations that shape a tradition, we can grasp those elements which characterise a style and define its identity. However, this undoubtedly perceptible, although fleeting and elusive note, remains to be heard, which alone makes it possible to put an individual work into perspective and grasp its meaning.

One tradition can then be distinguished from others. For example, we can distinguish the tradition of the method of indivisibles from the other traditions of infinitesimal mathematics of the 17th century or that of the marginalists from other economic traditions of the 19th and 20th centuries (such as those of Karl Marx or Alfred Marshall).

Through style, one can also isolate different currents within the same tradition (Bonaventura Cavalieri and Gilles de Roberval, in the first case; William Stanley Jevons and Léon Walras in the second case), or in a single work, when one identifies the traces of different traditions. Thus, we avoid the analogy and the global viewpoint that crushes differences to see only similarities.

1. Granger's Definition of Mathematical Style

If we confine ourselves to the history of mathematics, Granger defines the concept of mathematical style in the following way:

> The style appears to us here on the one hand as a way of introducing the concepts of a theory, of connecting them, of unifying them; and on the other hand, as a way of delimiting what intuition contributes to the determination of these concepts (Granger, 1968, 20).

To illustrate this definition, he studied several geometric examples, which he calls by the names of individual mathematicians, each of whom embodies the relevant geometric style: Euclidean, Cartesian, Desarguian, Grassmannian. Naturally, the whole study is done with precision and talent.

I want to start with Granger's definition to pose the question of the plurality of styles. The question is whether a fundamental mathematical work is distinguished by a single style or by several styles, and, on the other hand, whether the unity of a subject of mathematics during its evolution lies in one style or is an outcome of several styles. I, therefore, pose this question of the plurality of styles, firstly, within the same mathematical work, secondly, through the study of the same problem over the centuries. For the fundamental work, I chose the *Sphaerica* of Menelaus of Alexandria, and for the problem, I took the *isoperimetric problem*. The subject of study is the circle and the sphere in both cases.

2. The Style of Menelaus's *Sphaerica*

Menelaus of Alexandria (c. 70–40 CE) wrote a treatise entitled *Sphaerica (Spherics)* following the model of Euclid's *Elements*, that is, by chaining the propositions in a rigorous logical order. In this work, he studies the geometry of the sphere *per se*, not only in the physical three-dimensional solid, as in the *Elements.* Menelaus, unlike Euclid, examines the intrinsic properties of a spherical surface. In other words, he studies the geometry of the sphere as a chapter of solid geometry that his predecessors developed. He mainly focuses on the properties of duality and polarity, properties that do not characterise plane figures.

In his study, Menelaus admits Euclid's axioms and postulates, except for the fifth, the Parallel Postulate. He adopts the Euclidean definitions of geometric concepts, i.e., the definitions of the sphere, its centre, its circles, its diameters, its poles, etc., and adds three new definitions: those of the *spherical triangle*, the *quadrangle figures* on the sphere, and the *right*, *acute* and *obtuse angles* on the sphere.

If the definition of Granger already mentioned is invoked in this regard, the "way of introducing the concepts of a theory" is here quite different from that of Euclid, for, in this new geometry, the sum of the angles of any spherical triangle is greater than two right angles, disjoint lines do not exist, and any two lines intersect into two points. This case is certainly no longer the "style" of Euclidean plane geometry nor that of the stereometric books of Euclid's *Elements.*

From a mathematical point of view, this geometry is intimately linked to hyperbolic geometry, invented more than a millennium and a half later. It took centuries for the spherical geometry to be grounded on a system of axioms.

If we now consider the second part of Granger's definition, "a way of delimiting what intuition contributes to the determination of these concepts," again the style of Menelaus proceeds from that of Euclid: in fact, Menelaus banishes from spherical geometry the use of demonstration by *reductio ad absurdum* and keeps only the direct demonstration; moreover, he rejects the Euclidean demonstration by *application of areas*.

In short, in Menelaus's spherical geometry, we face the first non-Euclidean geometry, built from Euclid's axiomatic but excluding the Parallel Postulate and two methods of demonstration. Is it possible to discuss the Euclidean style, appropriately identified by Granger with the plane-geometric Books of the *Elements?* Certainly not. However, to describe the style of Menelaus as non-Euclidean would be a little stretched since he preserves what Euclid had abandoned. Thus, we face a mixture of two styles, a combination of the Euclidean style with some variety of non-Euclidean style. It could have been the first intuitionist style if Menelaus had always derived his constructions solely from the definitions. However, he sometimes does things differently, for example, in the first Proposition of his Book, which deals with the construction of an angle equal to a given angle. Unlike most of the propositions in Menelaus's *Sphaerica,* this proposition is proven based on Euclid's solid geometry; therefore, it is not a demonstration of spherical geometry. The reason is that Menelaus gives the Euclidean definition of the angle: the angle between two sides of a triangle at the top is a dihedral angle formed by the two planes that contain both sides; this requires starting with a Euclidean-style construction. So we cannot talk about a single style but a mixture of two styles, the second of which is not still perfectly advanced. This combination is imposed by the nature of the object studied by Menelaus: the sphere, regardless of the physical space.

3. The Question of the Plurality of Styles

I now turn to the second part of the question of plurality of styles, to the problem of the conceptualisation of the style of the same object throughout history. This time I borrow my example from geometry to stay close to Granger's choices. The examination of the isoperimetric problem allows us to see the succession of several styles, which fit together during the study of the same mathematical subject.

There are several reasons for the choice of the isoperimetric problem:

1. As already said, this problem belongs to the area from which Granger chooses his examples.
2. It concerns an ancient problem, like the example of Euclid chosen by Granger.
3. This is an example of the search for extremity values, hence its difficulty.

In a word, it is about to show that, of all the plane figures of a given perimeter, the circle—that is, the disk—has the greatest area; and that, of all the solids with the same total surface area, it is the sphere that has the greatest volume.

3.1. The Cosmological Style

At first glance, the search for extremity values was interesting to astronomers. They needed them to establish the sphericity of the heavens and the size of the world, to show the absolute perfection of their form. Mathematicians were called to demonstrate these properties and establish this cosmological fact. Moreover, this proposition about the circle and the sphere is intuitively evident, so that it may seem pointless to give a demonstration. However, the "delimitation of the intuitive contribution in the determination of concepts," as Granger says, has proven to be a very long and challenging task. I will outline his achievements briefly.

In any case, the problem of isoperimetric and isepiphanic figures appears to have a long history related to the cosmological perspective; this perspective made the problem perpetual and fruitful. Its wide diffusion is undoubtedly due to the revival of the first book of the *Almagest* and its commentary by Theon of Alexandria.

Ptolemy presents as an achievement of geometry the following result:

> Since, among different figures with equal perimeter, those with more sides are greater, the circle is the greatest of the plane figures, and the sphere is the greatest of the solids, and the heavens are the greatest of the bodies (Heiberg 1898, 13, lines 16-19).

However, he provides no proof. The commentators of the *Almagest*, since Theon of Alexandria, could no longer ignore such a formula without providing proof. Other mathematicians have shown interest in this problem, such as Heron of Alexandria and Pappus of Alexandria, in the fifth book of *The Collection.*[1]

Two relatively late testimonies agree on the attribution of the study of this problem to Zenodorus.

The first testimony comes from Theon of Alexandria, who states:

> We will prove this in an abbreviated way, drawn from Zenodore's demonstrations in his treatise *On Isoperimetric Figures* (Περὶ ἰσοπεριμέτρων σχημάτων) (Théon 1936, 33).

The second comes from Aristotle's commentator, Simplicius, who writes:

> It has been demonstrated, at least before Aristotle, whether it is true that he uses it as a proven truth, and by Archimedes, and in more detail by Zenodorus that among the isoperimetric figures the greatest is, among the plane figures, the circle, and among the solid figures, the sphere (Heiberg 1894, 412, lines 12-17).

Traces of the study of isoperimetric figures in Aristotle or Archimedes were searched for in vain. Simplicius agrees with Theon in attributing Zenodorus of the first extensive study. Zenodorus lived, most probably, after Archimedes and before Pappus and Theon; he must have lived between the 2nd century BC and the first half of the 4th century. Pappus (first half of the 4th century) quotes the first proposition from Zenodorus's book, and Theon (second half of the same century) summarises this book. However, the inaccuracy concerning Zenodorus's life dates prevents us from knowing with certainty whether the latter had written his treatise to justify Ptolemy's not yet demonstrated assertion.

3.2. Al-Khāzin's Geometric Style

Theon's text, which reports Zenodorus's results, and the *Almagest* were known in their Arabic translation by the 9th-century mathematicians and astronomers of Baghdad who initiated a new tradition of geometric research, notably by the philosopher and scholar al-Kindī. However, al-Khāzin and Ibn al-Haytham are recognised today as the leading representatives of this tradition. (Rashed 1993). The analysis of the works of these two mathematicians reveals a great distance between them.

[1] Cf. P. Ver Eecke's translation (1933, I, 239 sq.).

Al-Khāzin is a mathematician of the first half of the 10th century. He is known for his work in algebra and the Diophantine analysis for integers. He also starts with Ptolemy's quotation in his study of isoperimetric and isepiphanic figures. He proposes establishing Ptolemy's result not by computation but by using geometry. The guiding idea, of which al-Khāzin seems to be fully aware, is that of all convex figures of a given type (triangles, parallelograms, rhombuses, etc.), the more symmetrical one assumes an extremum for a certain magnitude (area, area ratio, perimeter, etc.). He proceeds in the following way: he fixes a parameter and varies the figure keeping it symmetrical about a definite straight line. Thus, by fixing the parallelogram's perimeter, he transforms it into a rhombus, keeping it symmetrical about its diagonal; the area increases in the process. With the help of several lemmas, al-Khāzin establishes the isoperimetric property of the regular polygons before finally passing over the theorem on a circle. He then shows the isepiphanic property with the help of regular polyhedra:

> Of all the convex solids with the same area, the sphere is the one with the greatest volume (Rashed 1996, 798).

We view in al-Khāzin two transformations: one is that of the object, the other is that of the style. Henceforth, the circle does not belong to the domain of plane geometric figures but falls under a class of them: the class of convex figures. Similarly, the sphere belongs to the class of convex solids. The style is no longer geometric in the broad sense, but it focuses on the inequalities necessary to research the geometry of convex domains. This research on the properties of convex figures will be one of the main themes of this subject throughout its history.

3.3. Ibn al-Haytham's Infinitesimalistic Style

About half a century later, the mathematician Ibn al-Haytham (d. after 1040) devoted a voluminous treatise to this problem. This treatise belongs to a series of works on the quadrature of curved surfaces and the cubature of solid curves. The mathematical context is no longer the same: it is shifted to the extremal properties of which Ibn al-Haytham was interested and, to study them, he combines infinitesimal methods and methods of projections. He departs from his predecessor in search of a "dynamic" demonstration. He then wrote his treatise on isoperimetric figures, which was at the forefront of contemporaneous mathematical research and for the following several hundred years.

Ibn al-Haytham begins with a quick examination of the case of plane figures. Just like his predecessor al-Khāzin, he compares regular polygons of the same perimeter and several different sides and demonstrates that

i. There are two regular polygons of the same perimeter; the one with the greatest number of sides has the greatest area.
ii. If a circle and a regular polygon have the same perimeter, then the area of the circle is greater than that of the polygon.

Unlike all his predecessors, Ibn al-Haytham uses the first property to establish the second, considering the circle as the limit of a sequence of regular polygons. He uses the properties of the upper bound; it is in this that his approach is "dynamic." It is noteworthy that in his demonstration, he assumed the existence of the boundary—the area of the disc—which Archimedes obtained in his *Measurement of a Circle*.

The second part of his treatise is devoted to isepiphanic figures. It opens with ten lemmas, which constitute the first proper treatise in the history of mathematics on the solid angles, which I will pass over in silence. In any case, these lemmas allow him to establish the following two propositions:

1. Of two regular polyhedra with similar faces and the same total area, the one with the greatest number of faces has the greatest volume.
2. Of two regular polyhedra with similar regular polygon faces inscribed in the same sphere, the one with the greatest number of faces has a greater area and greater volume.

Therefore, we observe that Ibn al-Haytham starts from the regular polyhedra. The two propositions I have just mentioned apply only to the case of tetrahedron, octahedron, and icosahedron since the number of faces of a regular polyhedron with square or pentagonal faces is fixed (6 or 12). However, Ibn al-Haytham's intention is clear from the above: from the comparison between polyhedra of the same area and a different number of faces, establishes the extremity of the sphere, i.e., approaches the sphere as the limit of a sequence of inscribed polyhedra. Nevertheless, this dynamic approach clashes with the finitude of the number of regular polyhedra, and I claim this fact remains incomprehensible on the part of a great mathematician, who knew Euclid's *Elements* better than anybody else. Nevertheless, this failure is compensated by a great success: the solid angle theory.

Ibn al-Haytham's treatise is far from the two previous styles, the cosmological and the geometric. Moreover, Ibn al-Haytham undertakes another study on the extremities in this new spirit. He compares different convex curves in a circular segment, considering that the length of each curve is the upper bound of the inscribed polygons, thus reducing the comparison between the curves to that between the polygons.

With Ibn al-Haytham, the extremal properties of figures and solids are studied, to which are now added those of the curves. The style changes accordingly and becomes infinitesimalistic on convex objects.

3.4. The Style of the Calculus of Variations

Going even further than Ibn al-Haytham was not possible until the foundation and the rise of differential calculus at the very end of the 17th century and the beginning of the 18th century, or more precisely with the first steps of the *calculus of variations*. The isoperimetric problem will continue to change form and become a problem for finding a curve, or a family of curves, that makes maximum or minimal the magnitude associated with each curve of a given set of curves. This problem started with Johann Bernoulli's (1667–1748) challenge of the mathematicians in June 1696 in a form that reproduces the famous *brachistochrone problem*:

> Given two points *A* and *B* in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at *A* and reaches *B* in the shortest time? (Bernoulli 1696, 269)[2]

Jacques Bernoulli had shown in 1697 that this curve is a *cycloid* (Bernoulli 1697, 211).

The isoperimetric problem is better studied on a different ground than the original cosmological perspective. This latter approach had run out, as we showed with al- Khāzin and transformed with Ibn al-Haytham. With the Bernoulli brothers, it is already a problem of calculating variations that their successor, Euler and afterwards Lagrange, will establish. Indeed, the study of

[2] "Datis in plano verticali duobus punctis *A* et *B*, assignare mobili *M* viam *AMB*, per quam gravitate sua descendens, et moveri incipiens a puncto *A*, brevissimo tempore perveniat ad alterum punctum *B*." (Given in a vertical plane two points *A* and *B*, assign to the moving [body] *M*, the path AMB, by means of which—descending by its own weight and beginning to be moved [by gravity] from point *A*—it would arrive at the other point *B* in the shortest time).

the preceding problem and those investigated by the calculus of variations led to differential equations for each problem found by Euler. The latter attempted to resume the problems and unify the methods of solution. Thus, the considered problem appears to require the determination among the curves the length $L = \int \sqrt{1+y'}dx$ for which the area $\int ydx$ is maximal (Euler 1744). However, an extremum may not be found when none of the curves of the solution gives an extremum. The difficulty raised by the existence of extremum will accompany the calculus of variations over a long period of its subsequent history.

3.5. The Style of Synthetic Geometry

Since the end of the 17th century and the 18th century, the isoperimetric problem has been studied using variational methods, such as Euler, Lagrange, and others.

A return to geometric methods was made from the beginning of the 19th century with Jakob Steiner (1796–1863), who introduced a geometric construction known as *Steiner symmetrisation*.

Going back to the original text:

For each different area of the circle and each direction of the line, a new smaller isoperimetric area is associated. These are geometric constructions in which, starting from a figure that is not a circle, one associates either a figure of the same perimeter but of the larger area, or a figure of the same area but of the smaller perimeter; the area and the perimeter of the circle remain invariant by these constructions. Steiner concludes that the theorem is proved for the circle, i.e., that, among the curves that enclose a given area, the circle has the smallest perimeter.

Let L be the perimeter of a closed curve in the plane and S the area it contains; then the isoperimetric problem requires to

Determine among all closed curves of length L the one with the greatest area and show that the solution is the circle.

The *isoperimetric deficit* of a curve is defined by the ordinary inequality:

$$\frac{L^2}{4\pi} - S \geq 0 \qquad (*)$$

and it is shown that equality is valid only for the circle.

Steiner (1971) gives five demonstrations, but every time he assumes the existence of an extremum. This demonstration means it implicitly assumes that, in all isoperimetric figures, there is one that has the maximum area.

With Steiner, the isoperimetric problem, such as the isepiphanic problem, can be expressed by isoperimetric inequalities like (*). His research aims to give basic demonstrations of these inequalities without assuming or demonstrating the existence of a maximum figure. He achieved this goal by improving the isoperimetric inequalities, that is, by showing that in the second member of inequality, where there is zero, a positive quantity can be substituted in general and that it can be cancelled only in the cases of the circle or the sphere. This process also avoids the notion of limit, except in defining the figures' perimeter, area, and volume.

The style is now that of synthetic geometry.

Following Steiner, in 1905, Felix Bernstein (1878–1956) demonstrated other inequalities, and Danish mathematician Tommy Bonnesen (1873–1935) published a book entitled *Les Problèmes des Isopérimètres et des Isépiphanes* (Bonnesen 1929) in which he demonstrated inequalities such as:

$$\frac{L^2}{4\pi} - S \geq \left(\frac{\pi}{4}\right)\left(R - r\right)^2$$

where R and r are the rays of the greatest circles, respectively circumscribed to and inscribed in the convex curve L. We immediately see that if $R = r$, we have equality for the circle.

As can be seen, the isoperimetric problem, in a way at the origin of the calculation of variations at the beginning of the 18th century, became the object of the theory of convex domains on the plane or space, and convex curves,[3] from the end of the 19th century and the beginning of the following century. Thus, from the end of the 19th century, the isoperimetric problem changed its scope: it now consists of determining, among all the closed plane curves of a given perimeter, the one that contains the greatest area. This same problem can still be followed in other fields of recent geometry, where the inequalities were found to serve in one way or another. This long and rich history illustrates the variety of styles encountered in the conceptualisation of the same problem.

[3] "Convex domain" on the plane or space is taken to mean a set of points such that given any points *A* and *B*, it contains the whole line segment *AB* that joins them. The boundary of a convex and bounded figure is a closed convex curve.

Conclusion

In conclusion, it seems that it is clear from the example of Menelaus, the founder of spherical geometry, that the multiplicity of styles is the effect of the gestation of a new style, which cannot exist without the old one. We have observed in this example that the Euclidean style, defined from the axiomatics of Euclid's *Elements* and the theory of proportions, was called upon to deal with a new object that does not admit a postulate essential to the definition of this style, and which even excludes the means of conceptualisation of Euclidean geometry. Menelaus had to combine the Euclidean style with another style, which can be described as a proto-intuitionist. This intersection between two styles is not uncommon in the founding works of new mathematical disciplines: it can be observed in the *Conics* of Apollonius, the *Optics* of Ptolemy, and other works.

As for the example of the isoperimetric problem, it seems that the multiplicity of styles is due to the transformation of the object of research, aroused by the ontological density of the circle and the sphere, whose properties are inexhaustible. The multiplication of styles that involve the languages of cosmology, the geometry of figures and solid convexes, infinitesimal geometry, differential, and integral calculation, metric geometry of convex domains, is the effect of the acquisition of other methods, forged in the event of new research in other fields, and which have allowed the unveiling of new layers in the thickness of the objects—the circle and the sphere. Beyond the plurality of styles and methods, the unifying element of this subject lies in the constant effort to determine the extremality properties of certain convex domains and develop a theory of these domains.

This long and rich history also illustrates what we already learned by the example of the *Sphaerica*: the multiplicity of styles is the hall of mathematical research that deals with dense and fruitful objects. One might dare to say that the uniqueness of the style is an indication of the lightness or even the poverty of the object.

Perhaps this is why Granger proposed this heuristic instrument to philosophers and historians of science, which allows us to marry this complex dialectics between uniqueness and multiplicity, which stirs many mathematics and science subjects.

Bibliography

1. Bernoulli Johann, (1696), "Problema novum ad cujus solutionem Mathematici invitantur" (A new problem to whose solution mathematicians are invited), *Acta Eruditorum*, 18 (June), p. 269.
2. Bernoulli Johann (1697), "Curvatura radii in diaphanis non uniformibus, Solutioque Problematis a se in Actis 1696, p. 269, propositi, de invenienda Linea Brachystochrona, id est, in qua grave a dato puncto ad datum punctum brevissimo tempore decurrit, & de curva Synchrona seu radiorum unda construenda" (The curvature of [light] rays in non-uniform media, and a solution of the problem [which was] proposed by me in the *Acta Eruditorum* of 1696, p. 269, from which is to be found the brachistochrone line [i.e., curve], that is, in which a weight descends from a given point to a given point in the shortest time, and on constructing the tautochrone or the wave of [light] rays.), *Acta Eruditorum*, 19 (May), pp. 206-211.
3. Bonnesen Tommy (1929), *Les Problèmes des Isopérimètres et des Isépiphanes*, Paris, Gauthier-Villars.
4. Euler Leonhard (1744), *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes* (A method for finding curved lines enjoying properties of maximum or minimum, or solution of isoperimetric problems in the broadest accepted sense), Lausanne, Lausanne & Geneva: Marcum-Michaelem Bousquet, Volume 1744, pp. 1-322, *Opera Omnia* Series 1, Volume 24, pp. 1-308.
5. Granger Gilles Gaston (1987), *Essai d'une philosophie du style*, Paris: Armand Colin, 1968, 2e éd., Odile Jacob.
6. Heiberg Johan Ludvig (1898), *Claudii Ptolemaei opera quae extant omnia. I. Syntaxis mathematica*, Leipzig.
7. Heiberg Johan Ludvig (1894), *Simplicii in Aristotelis De Caelo commentaria*, Berolini: G. Reimeri.
8. Rashed Roshdi (1996), *Les Mathématiques infinitésimales du IXe au XIe siècle*, Vol. I: Fondateurs et commentateurs: *Banū Mūsā, Thābit ibn Qurra, Ibn Sinān, al-Khāzin, al-Qūhī, Ibn al-Samḥ, Ibn Hūd*, London: al-Furqān Islamic Heritage Foundation.
9. Rashed Roshdi (1993), *Les Mathématiques infinitésimales du IXe au XIe siècle*, Vol. II: *Ibn al-Haytham*, London: Al-Furqān.
10. Steiner Jakob (1971), „Einfache Beweise der isoperimetrische Hauptsätze", *Gesammelte Werke*, NY, vol. II, pp. 285-308.
11. Théon d'Alexandrie (1931, 1936, 1943), *Commentaires de Pappus et de Théon d'Alexandrie sur l'Almageste*, Books I-III, ed. A. Rome, Vatican City: Billiotheca Apostolica Vaticana, Studi e Testi, No. 72, 106, and 54, respectively. Theon's commentary on the Almagest is in Books II-III.
12. de ver Eecke Paul (1933), *La Collection mathématique*, Paris et Bruges.